Knots and Some Chess

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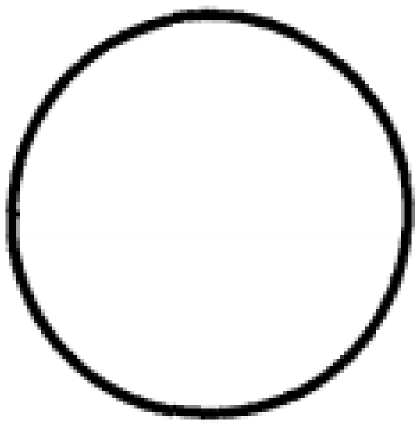
In DRP this semester, I read The Knot Book, by Collin Adams. Here are a couple things I have learned. At the end, I have applied them to calculate linking numbers of graphs generated from famous chess games.

Introduction:

A knot is defined as a topological circle embedded into 3 dimensional space. This means it is a closed curve that does not intersect itself at any point. Knot diagrams of some simple knots are shown below:



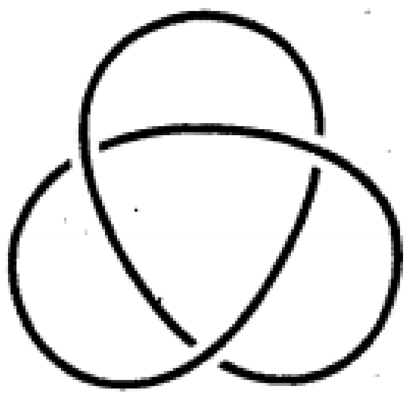


Figure 1: unknot (left), trefoil knot (right) (1)

The same knot can be embedded in different ways, for example, the two knots below are actually the exact same knot.

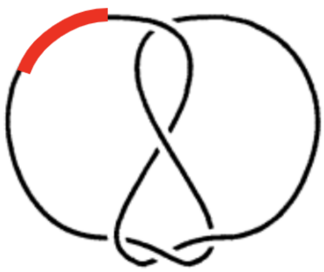


Figure 2: the same knot embedded in two ways (1)

In this case, it is not too complicated to see that these two knot diagrams come from the same knot - just take the section highlighted in red, twist it once, and move it to the corresponding location in the second knot. However, for many knot diagrams it is not immediately apparent that knots are equivalent. For a more rigorous approach to proving equivalence, Reidemeister moves can be used:

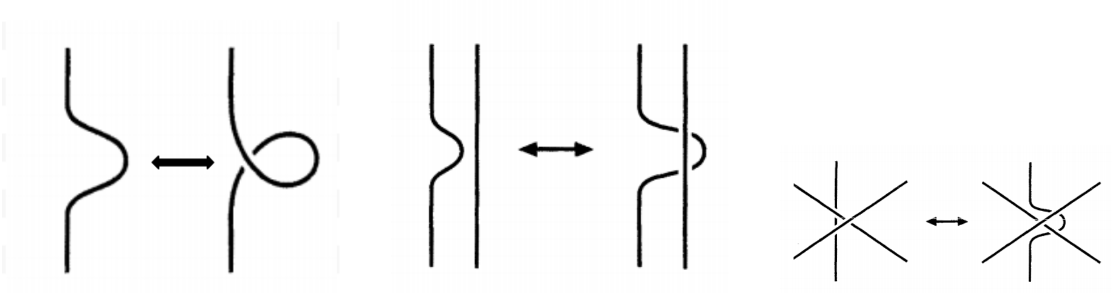


Figure 3: from left to right - a type 1, type 2, and type 3 Reidemeister move (1)

If there exists a sequence of these transformations under which one knot diagram can be transformed to another, then the two are projections of the same knot.

Diagrams are not limited to only one topological circle - when multiple circles are included, we have a link:

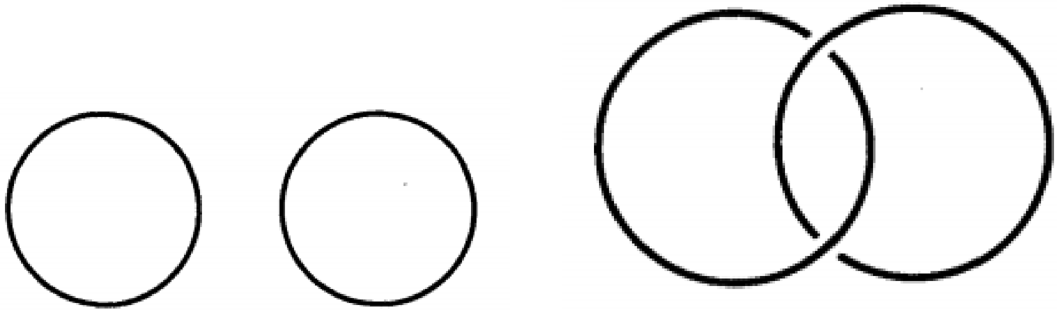


Figure 4: the unlink (left) and the Hopf link (right) (1)

Finally, let's define a graph as a set of points in 3 dimensional space connected by edges. One type of graph is known as a planar graph, which lies in a 2 dimensional plane:

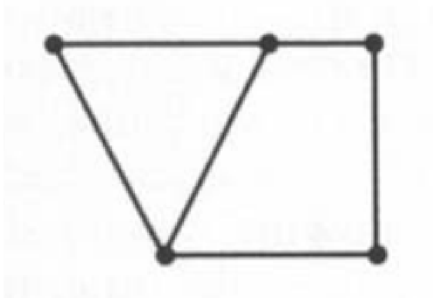


Figure 5: A planar graph (1)

Knots and links can also be embedded in graphs:

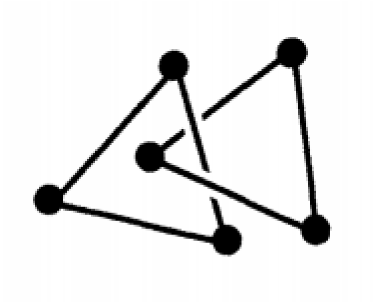


Figure 6: a link embedded within a graph projection (1)

A particularly interesting example of this is found in the graph K_6:

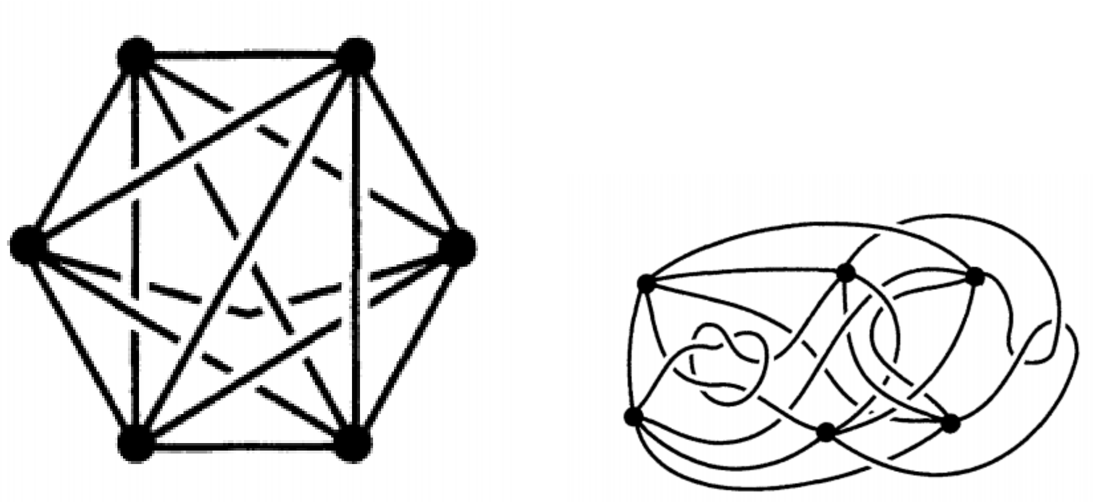


Figure 7: two embeddings of the graph K_6 (1)

K_6 is defined as a set of 6 vertices, each of which are connected by an edge to every other vertex. From here, let's define a triangle as a set of three points connected to each other by edges. No matter how K_6 is embedded in space, it will contain at least one pair of linked triangles, like the ones shown in figure 6. Thus, we can say K_6 is intrinsically linked.

For the next part, I have used a couple famous chess games to generate graph diagrams and check if they are intrinsically linked. I've adopted a method to create the graphs. I played through the game from the beginning, and at any point where a piece was captured, a vertex was placed at both the square on which the capturing piece was prior to the move and on the square it was afterward. I hoped to see whether differences in pawn structure, which refers to the arrangement of pawns on the board, castling directions, which refer to a special move players can do which moves their king two squares in one horizontal direction and their rook next to the king such that it switches to the opposite side in relation to it, and game length would have any impact on the number of links formed. The choice of games was based on these properties, as well as my own preference. Here is this method applied to a relatively short (22 move) game.

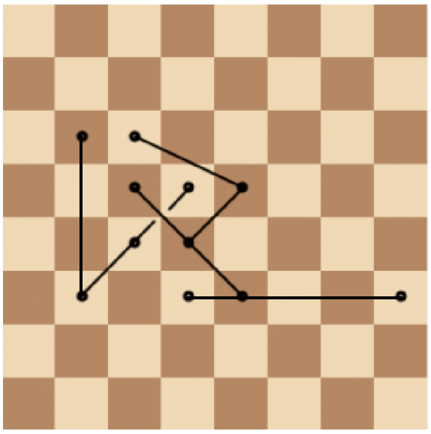


Figure 8: A graph diagram generated from the chess game: Gerasimov - Smyslov, 1935

Slightly left of center, there is an intersection between edges. Another convention I used was that the edge that was placed first (the move which came first) was placed on top. This was the first game I tried this method on, but since there is nothing resembling a topological circle here, there is no way to generate interesting information about linking from this game.

I chose Garry Kasarov's "immortal game" as the next game because it was of fairly average length (44 moves), and for the most part was played in a very "open" pawn structure position, meaning pawns were traded in the center, which allows for quite a bit more freedom of movement for pieces. I anticipated this would increase chances for linking as edges could be created more freely in the center, where chances are the highest for them to be intersected, potentially creating links. Before moving on to the diagram of this game, I would like to define one more term: a cycle is a set of vertices connected by edges such that you could trace a path from one vertex back to itself without crossing the same edge twice. In the following game, a few cycles appeared, and I have highlighted them:

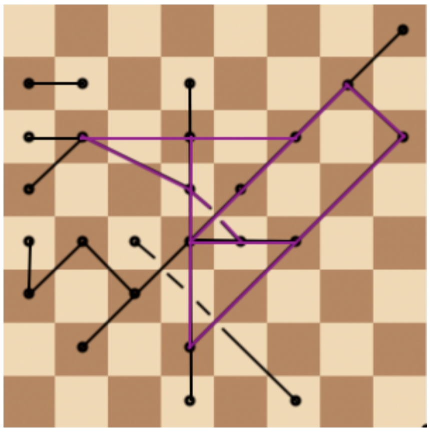


Figure 9: A graph diagram generated from Garry Kasparov's "Immortal Game" against Veselin Topalov - 1999

Right in the center of the board, one of the cycles intersects with two other cycles, let's isolate those intersections:

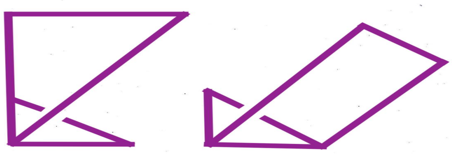


Figure 10: isolated cycles from figure 9

To create links out of subsections such as these, the edge that these two cycles share must be split into two - there are four ways to do this.

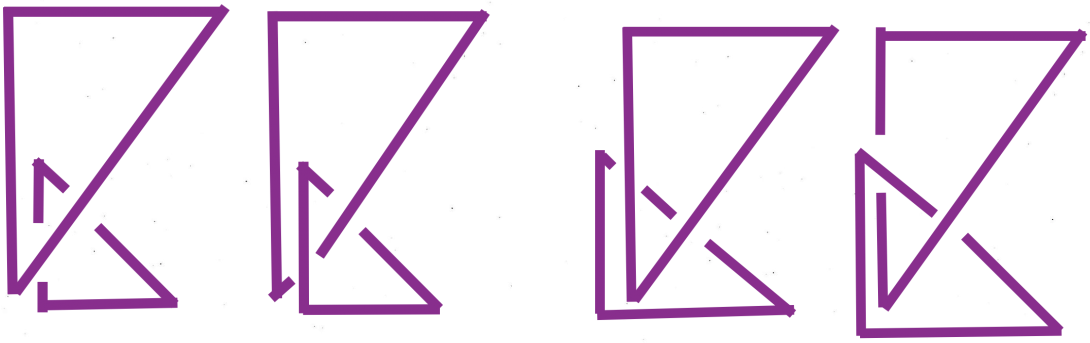


Figure 11: possible ways of resolving edges - the first and third are unlinks, while the second and fourth are Hopf links

It turns out that when this convention is applied to any two cycles that intersect and share an edge, two out of the four possible resolutions contain links. Now let the linking number of each of these diagrams be equal to 1 if there exists a Hopf link and equal to 0 if there exists an unlink. Therefore, the average linking number of all the possible resolutions is 1/2. Since this situation occurs twice in this game's graph diagram, this game has an average linking number of 1. Here are the graph diagrams of some more games:

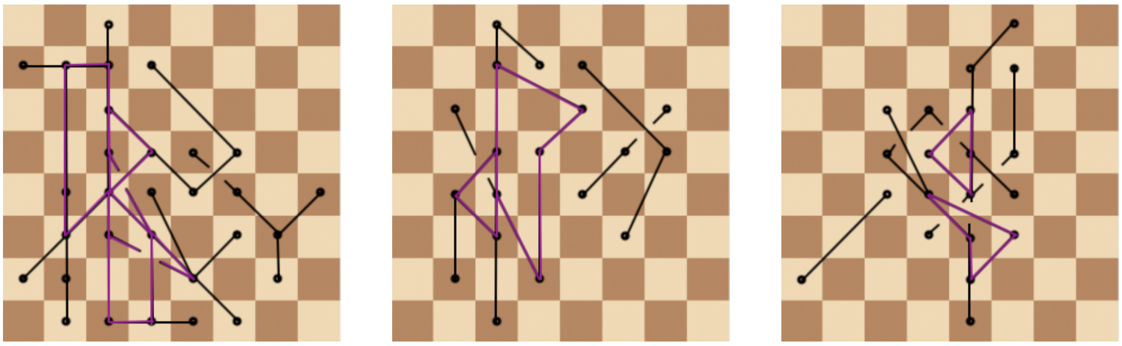


Figure 12: from left to right: Mikhail Botvinnik - Mikhail Tal, World Championship Game 6, 1960; Vishwanathan Anand - Magnus Carlsen 2007; Mikhail Tal - Bent Larsen 1965

From left to right, the graph diagrams had average linking numbers of 2, 0, and 0. I expected "closed" pawn structure positions, or positions where pawns lock up in the center instead of trading to generally have the lowest linking numbers. Botvinnik - Tal was played in a notoriously closed chess opening, the King's Indian Defense, yet actually had the highest linking number out of all the games I looked at. I overlooked the impact of the higher usage of knights, which are pieces that move two squares in a certain direction and one more in a perpendicular direction, in these types of positions, since edges formed by knight moves cannot pass through a preexisting vertex and therefore have the highest likelihood of creating a crossing. While this was surprising, the second game, Anand - Carlsen, was also very closed and generated a linking number of 0. For the last game, Tal - Larsen, I wanted to test the effect of players castling in opposite directions. These positions generally mean each player focuses on a different side of the board, and I anticipated these would have low linking, which this game's graph diagram's average linking number of 0 was consistent with.

Out of the three factors I looked at, the one impacting average linking number the most seemed to be game length:

Game	Length (moves)	Average linking number
Gerasimov - Smyslov	22	0
Tal - Larsen	37	0
Anand - Carlsen	38	0
Kasparov - Topalov	44	1
Tal - Botvinnik	47	2

Figure 13: game length and average linking number

Citations:

1) Adams, C. C. (2010). *The knot book: An elementary introduction to the mathematical theory of knots.* Providence, RI: American Mathematical Society.